On a class of inhomogeneous extensions for integrable evolution systems*

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Abstract

In the present paper we prove the integrability (in the sense of existence of formal symmetry of infinite rank) for a class of block-triangular inhomogeneous extensions of (1+1)-dimensional integrable evolution systems. An important consequence of this result is the existence of formal symmetry of infinite rank for "almost integrable" systems, recently discovered by Sanders and van der Kamp.

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Introduction

It is well known that the existence of infinite number of generalized (or higher order) symmetries for a system of PDEs is one of the most important signs of its integrability, see for example [4, 10, 11, 12, 13]. Moreover, for a long time it was generally believed that the existence of only one nontrivial local generalized symmetry implies the existence of infinitely many such symmetries, cf. [4].

However, the latter statement is not true, as shows the example of Bakirov system [1]

$$\begin{aligned} \partial u/\partial t &= \partial^4 u/\partial x^4 + v^2, \\ \partial v/\partial t &= (1/5)\partial^4 v/\partial x^4. \end{aligned} \tag{1}$$

This system has only one non-Lie-point x, t-independent local generalized symmetry, as it was proved by Beukers, Sanders and Wang [2] using the sophisticated methods of number theory. What is more, the situation remains unchanged even if we pass to x, t-dependent local generalized symmetries, see [14].

Sanders and van der Kamp [8] have generalized this result and found a counterexample to the conjecture of Fokas [4] stating that if an s-component system of PDEs has s non-Lie-point local generalized symmetries, then it has infinitely many symmetries of this kind. Namely, they have exhibited a two-component evolution system possessing only two non-Lie-point x, t-independent local generalized symmetries. This system is of the form

$$\begin{aligned} \partial u/\partial t &= a u_7 + b v_1 v_2 + 7 v v_3, \\ \partial v/\partial t &= v_7, \end{aligned} \tag{2}$$

where $a = -(42\alpha^5 + 280\alpha^4 + 700\alpha^3 + 798\alpha^2 + 504\alpha + 104)$, $b = 7\alpha^5 + 49\alpha^4 + 133\alpha^3 + 175\alpha^2 + 126\alpha + 56$, and α is a root of the equation $\alpha^6 + 7\alpha^5 + 19\alpha^4 + 25\alpha^3 + 19\alpha^2 + 7\alpha + 1 = 0$, $u_i = \partial^i u/\partial x^i$, $v_i = \partial^i v/\partial x^i$.

Let us note that both systems (1) and (2) are exactly solvable. Indeed, one can find the general solution of the second equation for v, then plug it into the first equation and find its general solution for u.

Since the systems (1) and (2) possess only a finite number of non-Lie-point local generalized symmetries and at the same time are exactly solvable, it is interesting to find out whether they pass or fail other integrability tests. One of the most powerful and algorithmic tests of this kind is the existence of nondegenerate formal symmetry of infinite rank and nonzero degree, see [10, 11, 12, 13] and Section 1 below for details. For the Bakirov system the existence of formal symmetry with these properties was proved by Bilge [3]. It is natural to ask whether a similar result could be established for the system (2), as well as for other systems listed in [8].

In the present paper we show that this is indeed possible for quite a large class of evolution systems of the form (5), which naturally generalize the Bakirov system [1], and those of Sanders and van der Kamp [8], see Proposition 1 and Corollaries 3–5 below for details. Note that, in the terminology of Kupershmidt [9], the system (5) can be thought of as a particular case of *inhomogeneous nonlinear extension* of its last block, that is, $\partial \vec{u}^s/\partial t = \vec{f}^s(x, t, \vec{u}^s, \vec{u}^s_1, \ldots, \vec{u}^s_n)$.

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1 Basic definitions and structures

Consider a $(1+1)$-dimensional evolution system

$$\partial \mathbf{u}/\partial t = \mathbf{F}(x, t, \mathbf{u}, \ldots, \mathbf{u}_n) \tag{3}$$

for the q-component vector function $\mathbf{u} = (u^1, \ldots, u^q)^T$. Here $\mathbf{u}_j = \partial^j \mathbf{u}/\partial x^j$, $\mathbf{u}_0 \equiv \mathbf{u}$, $\mathbf{F} = (F^1, \ldots, F^q)^T$, and the superscript T denotes the matrix transposition. In what follows we assume that $n \geq 2$ and $\partial \mathbf{F}/\partial \mathbf{u}_n \neq 0$.

Let us recall that a function f of $x, t, \mathbf{u}, \mathbf{u}_1, \ldots,$ is called *local* (cf. [12]) if it depends only on a finite number of variables $\mathbf{u}_j$. The operators of total derivatives with respect to x and t on the space of (smooth) local functions take the form $D_x \equiv D = \partial/\partial x + \sum_{i=0}^{\infty} \mathbf{u}_{i+1} \partial/\partial \mathbf{u}_i$ and $D_t = \partial/\partial t + \sum_{i=0}^{\infty} D^i(\mathbf{F}) \partial/\partial \mathbf{u}_i$, cf. [11, 12, 13].

Consider [11, 12, 13] a formal series in powers of D of the form

$$\mathfrak{H} = \sum_{j=-\infty}^{q} h_j D^j,$$

where h_j are $(p \times p)$-matrix-valued local functions. The greatest $m \in \mathbb{Z}$ such that $h_m \neq 0$ is called the *degree* of $\mathfrak{H}$ and is denoted by $m = \deg \mathfrak{H}$. The formal series $\mathfrak{H}$ is called *nondegenerate*, if $\det h_m \neq 0$, $m = \deg \mathfrak{H}$. Following the usual convention [13], we assume that $\deg 0 = -\infty$.

A formal series $\mathfrak{R} = \sum_{j=-\infty}^{r} \eta_j D^j$, where η_j are $(q \times q)$-matrix-valued local functions, is called a *formal symmetry* of infinite rank (see [10, 11, 12, 13]) for (3), if it satisfies the equation

$$D_t(\mathfrak{R}) = [\mathbf{F}_*, \mathfrak{R}]. \tag{4}$$

Here we set $D_t(\mathfrak{R}) = \sum_{j=-\infty}^{r} D_t(\eta_j) D^j$, $\mathbf{F}_* = \sum_{i=0}^{n} \partial \mathbf{F}/\partial \mathbf{u}_i D^i$, and $[\cdot, \cdot]$ stands for the usual commutator of two formal series: $[\mathfrak{A}, \mathfrak{B}] = \mathfrak{A} \circ \mathfrak{B} - \mathfrak{B} \circ \mathfrak{A}$.

The multiplication law $\circ$ (see for example [13]) is defined for monomials as

$$aD^i \circ bD^j = a \sum_{q=0}^{\infty} \frac{i(i-1)\cdots(i-q+1)}{q!} D^q(b) D^{i+j-q}, \quad i, j \in \mathbb{Z},$$

and is extended by linearity to the set of all formal series. In what follows we shall omit $\circ$ unless this leads to confusion. In particular, for $k \in \mathbb{N}$ we set $\mathfrak{R}^k = \mathfrak{R} \circ \mathfrak{R}^{k-1}$.

2 The main result

Consider an evolution system (3) of the form

$$\begin{aligned}
\frac{\partial \vec{u}^1}{\partial t} &= \vec{f}^1(x, t, \vec{u}^1, \vec{u}^1_1, \ldots, \vec{u}^1_n) + \vec{h}^1(x, t, \vec{u}^2, \vec{u}^2_1, \ldots, \vec{u}^2_{n-1}, \ldots, \vec{u}^s, \vec{u}^s_1, \ldots, \vec{u}^s_{n-1}), \\
\frac{\partial \vec{u}^2}{\partial t} &= \vec{f}^2(x, t, \vec{u}^2, \vec{u}^2_1, \ldots, \vec{u}^2_n) + \vec{h}^2(x, t, \vec{u}^3, \vec{u}^3_1, \ldots, \vec{u}^3_{n-1}, \ldots, \vec{u}^s, \vec{u}^s_1, \ldots, \vec{u}^s_{n-1}), \\
&\vdots \\
\frac{\partial \vec{u}^s}{\partial t} &= \vec{f}^s(x, t, \vec{u}^s, \vec{u}^s_1, \ldots, \vec{u}^s_n),
\end{aligned} \tag{5}$$

where $n \geq 2$, $\vec{u}^\alpha_j = \partial^j \vec{u}^\alpha/\partial x^j$, $\vec{u}^\alpha = (u^{\alpha,1}, \ldots, u^{\alpha,q_\alpha})^T$, $\vec{f}^\alpha = (f^{\alpha,1}, \ldots, f^{\alpha,q_\alpha})^T$, $\vec{h}^\alpha = (h^{\alpha,1}, \ldots, h^{\alpha,q_\alpha})^T$.

The system (5) is nothing but a particular case of (inhomogeneous nonlinear) extension of $\vec{u}^s_t = \vec{f}^s$, cf. [9]. It turns out that under some extra conditions the existence of formal symmetries of infinite rank and nonzero degree for the systems $\vec{u}^\alpha_t = \vec{f}^\alpha$, $\alpha = 1, \ldots, s$, implies the same property for the system (5) with arbitrary $\vec{h}^\alpha$.

In what follows we assume the ground field to be algebraically closed, so that any matrix can be brought into Jordan's normal form, see e.g. [6]. Then we have the following result.

Proposition 1 *Suppose that the matrices $\partial\vec{f}^\alpha/\partial\vec{u}_n^\alpha$ and $\partial\vec{f}^\beta/\partial\vec{u}_n^\beta$ have no common eigenvalues (i.e., the eigenvalues in question are distinct as functions) for all $\alpha\neq\beta$, $\alpha,\beta=1,\dots,s$, and at least one of these matrices is nonzero. Further assume that each of the evolution systems $\vec{u}_t^\alpha=\vec{f}^\alpha$, $\alpha=1,\dots,s$, has a formal symmetry $\mathfrak{L}_\alpha$ of infinite rank and nonzero degree, and the coefficients of $\mathfrak{L}_\alpha$ for $\alpha=1,\dots,s-1$ depend on x and t only.*

Then the system (5) with arbitrary (smooth) functions $\vec{h}^\alpha(x,t,\vec{u}^{\alpha+1},\vec{u}_1^{\alpha+1},\dots,\vec{u}_{n-1}^{\alpha+1},\dots,\vec{u}^s,\vec{u}_1^s,\dots,\vec{u}_{n-1}^s)$, $\alpha=1,\dots,s-1$, also possesses a formal symmetry of infinite rank and nonzero degree.

Moreover, if all $\mathfrak{L}_\alpha$ are nondegenerate, then (5) possesses a nondegenerate *formal symmetry of infinite rank and nonzero degree.*

Proof. Let us start with the following lemma (cf. [11, Proposition 2.1]).

Lemma 2 *Suppose that the matrices $\partial\vec{f}^\alpha/\partial\vec{u}_n^\alpha$ and $\partial\vec{f}^\beta/\partial\vec{u}_n^\beta$ have no common eigenvalues (i.e., the eigenvalues in question are distinct as functions) for all $\alpha\neq\beta$, $\alpha,\beta=1,\dots,s$, and at least one of these matrices is nonzero.*

Then there exists a unique formal series

$$\mathfrak{T}=\mathbf{1}+\sum_{i=-\infty}^{-1}T_iD^i$$

such that T_i are upper block-triangular $(q\times q)$-matrix-valued local functions with zero diagonal blocks and we have

$$\mathfrak{V}\equiv\mathfrak{T}\mathbf{F}_*\mathfrak{T}^{-1}+D_t(\mathfrak{T})\mathfrak{T}^{-1}=\mathrm{diag}(\mathfrak{F}_1,\dots,\mathfrak{F}_s).$$

Here $\mathbf{1}$ is a $q\times q$ unit matrix, $q=\sum_{\alpha=1}^s q_\alpha$, $\mathfrak{F}_\alpha=\sum_{i=0}^n\partial\vec{f}^\alpha/\partial\vec{u}_i^\alpha D^i$, and $\mathbf{F}$ stands for the right-hand side of (5).

Before we prove this lemma, let us apply it for the proof of Proposition 1. Equation (4) under the transformation $\mathfrak{R}\to\mathfrak{L}=\mathfrak{T}\mathfrak{R}\mathfrak{T}^{-1},\mathbf{F}_*\to\mathfrak{V}$ becomes

$$D_t(\mathfrak{L})=[\mathfrak{V},\mathfrak{L}]. \tag{6}$$

For the system (5), using the assumption that the matrices $\partial\vec{f}^\alpha/\partial\vec{u}_n^\alpha$ have no common eigenvalues, it is easy to check (cf. [11]) that the coefficients of any solution $\mathfrak{L}$ of (6) are block-diagonal matrices, i.e., $\mathfrak{L}=\mathrm{diag}(\mathfrak{R}_1,\dots,\mathfrak{R}_s)$, where $\mathfrak{R}_\alpha$ is a formal series whose coefficients are $q_\alpha\times q_\alpha$ matrices, and thus (6) is broken into s blocks:

$$D_t(\mathfrak{R}_\alpha)=[\mathfrak{F}_\alpha,\mathfrak{R}_\alpha]. \tag{7}$$

Each of equations (7) for $\alpha=1,\dots,s$ has a solution $\mathfrak{R}_\alpha=\mathfrak{L}_\alpha$. Thus, equation (6) has a solution $\mathfrak{L}=\mathrm{diag}(\mathfrak{L}_1,\dots,\mathfrak{L}_s)$, and $\mathfrak{R}=\mathfrak{T}^{-1}\mathfrak{L}\mathfrak{T}$, with $\mathfrak{T}$ constructed in Lemma 2, is a formal symmetry of nonzero degree and infinite rank for (5).

If all $\mathfrak{L}_\alpha$ are nondegenerate, then we can choose $\mathfrak{L}$ to be $\mathfrak{L}=\mathrm{diag}(\mathfrak{L}_1^{p_1},\dots,\mathfrak{L}_s^{p_s})$ and $\mathfrak{R}=\mathfrak{T}^{-1}\mathfrak{L}\mathfrak{T}$, where $p_\alpha=m/m_\alpha$, $m_\alpha=\deg\mathfrak{L}_\alpha$, and m is the least common multiple of m_α, $\alpha=1,\dots,s$. Thus constructed $\mathfrak{R}$ obviously will be a nondegenerate formal symmetry of infinite rank and nonzero degree m for (5). This remark completes the proof of Proposition 1. □.

Note that D_t in (7) *does not* coincide with the operator $\partial/\partial t+\sum_{i=0}^\infty D^i(\vec{f}^\alpha)\partial/\partial\vec{u}^\alpha$ (no sum over α). Therefore, if the coefficients of $\mathfrak{L}_\alpha$, $\alpha=1,\dots,s-1$, depend not only on x and t, there is no obvious way to construct the solutions of (7) for $\alpha=1,\dots,s-1$ and to extend the result of Proposition 1 to this case.

Proof of Lemma 2. By the above, $\mathfrak{T}$ is assumed to have the form

$$\mathfrak{T}=\begin{pmatrix}\mathbf{1}_{q_1} & \mathfrak{T}_{12} & \dots & \mathfrak{T}_{1s}\\ 0 & \mathbf{1}_{q_2} & \dots & \mathfrak{T}_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{1}_{q_s}\end{pmatrix},$$

where $\mathbf{1}_{q_\alpha}$ stands for $q_\alpha\times q_\alpha$ unit matrix, $\mathfrak{T}_{\alpha\beta}$ are formal series of degree not higher than -1: $\mathfrak{T}_{\alpha\beta}=\sum_{r=1}^\infty\tau_{\alpha\beta}^rD^{-r}$, and the coefficients $\tau_{\alpha\beta}^r$ are $q_\alpha\times q_\beta$ matrices.

It is clear that for $\mathbf{F}$ being the right-hand side of (5) $\mathbf{F}_*$ has a similar structure:

$$\mathbf{F}_* \equiv \mathfrak{V} + \mathfrak{B} = \begin{pmatrix} \mathfrak{F}_1 & \mathfrak{B}_{12} & \dots & \mathfrak{B}_{1s} \\ 0 & \mathfrak{F}_2 & \dots & \mathfrak{B}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathfrak{F}_s \end{pmatrix},$$

where $\mathfrak{B}_{\alpha\beta}$ are formal series of degree not higher than $n-1$:

$$\mathfrak{B}_{\alpha\beta} = \sum_{r=0}^{n-1} b^r_{\alpha\beta} D^r,$$

and the coefficients $b^r_{\alpha\beta}$ are $q_\alpha \times q_\beta$ matrices.

Multiplying the equality $\mathfrak{V} = \mathfrak{T}\mathbf{F}_*\mathfrak{T}^{-1} + D_t(\mathfrak{T})\mathfrak{T}^{-1}$ by $\mathfrak{T}$ on the right, we find

$$\mathfrak{V}\mathfrak{T} = \mathfrak{T}\mathbf{F}_* + D_t(\mathfrak{T}).$$

Inserting in this formula the expressions for $\mathfrak{T}$, $\mathfrak{V}$ and $\mathbf{F}_*$ and equating "blockwise" its left-hand side and right-hand side, we obtain identities of the form $\mathfrak{F}_\alpha = \mathfrak{F}_\alpha$ or $0 = 0$ together with the following equations:

$$\mathfrak{F}_\alpha \mathfrak{T}_{\alpha\beta} - \mathfrak{T}_{\alpha\beta}\mathfrak{F}_\beta = \sum_{\gamma=\beta+1}^{s} \mathfrak{T}_{\alpha\gamma}\mathfrak{B}_{\gamma\beta} + \mathfrak{B}_{\alpha\beta} + D_t(\mathfrak{T}_{\alpha\beta}), \quad s \geq \beta > \alpha \geq 1. \tag{8}$$

Provided the coefficients of formal series $\mathfrak{T}_{\alpha\gamma}$, $\gamma > \beta$, are known, we can find from (8) the coefficients of $\mathfrak{T}_{\alpha\beta}$, solving *algebraic* equations only. Indeed, equating the coefficients at D^{n-p} on the left- and right-hand side of (8) yields

$$a^\alpha_n \tau^p_{\alpha\beta} - \tau^p_{\alpha\beta} a^\beta_n = \eta^p_{\alpha\beta},$$

where $a^\alpha_n \equiv \partial \vec{f}^\alpha / \partial \vec{u}^\alpha_n$, and $\eta^p_{\alpha\beta}$ is a $q_\alpha \times q_\beta$ matrix whose entries are differential polynomials in the entries of the matrices $\tau^j_{\alpha\beta}$ with $j < p$, and in the entries of coefficients of the formal series $\mathfrak{F}^\alpha$, $\mathfrak{B}_{\alpha\gamma}$ and $\mathfrak{T}_{\alpha\gamma}$ with $\gamma > \beta$.

Since the matrices a^α_n have no common eigenvalues by assumption, we always can (see [6]) successively solve the above equations with respect to $\tau^p_{\alpha\beta}$ for $p = 1, 2, \dots$, starting with the equations for $\tau^p_{\alpha s}$ and using previously solved equations, if any occur. What is more, the solution to these equations is unique [6]. This completes the proof of the lemma. □

As an example, consider the system

$$\begin{aligned} u_t &= (1-c)u_3 + cv_3 + 3uu_1 + 3uv_1 + 3vu_1 + 3vv_1 + g(w, w_1, w_2), \\ v_t &= cu_3 + (1-c)v_3 + 3uu_1 + 3uv_1 + 3vu_1 + 3vv_1 + h(w, w_1, w_2), \\ w_t &= w_3 + ww_1, \end{aligned} \tag{9}$$

where c is a constant.

The system

$$\begin{aligned} u_t &= (1-c)u_3 + cv_3 + 3uu_1 + 3uv_1 + 3vu_1 + 3vv_1, \\ v_t &= cu_3 + (1-c)v_3 + 3uu_1 + 3uv_1 + 3vu_1 + 3vv_1, \end{aligned}$$

discovered by Foursov [5], possesses a degenerate formal symmetry of infinite rank

$$\mathfrak{L}_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} D^2,$$

and for $c \neq \frac{1}{2}$ can be written in bi-Hamiltonian form in infinitely many ways, see [5].

The equation $w_t = w_3 + ww_1$ is nothing but the fabulous KdV equation, which has a nondegenerate formal symmetry of infinite rank

$$\mathfrak{L}_2 = D^2 + \frac{2}{3}u + \frac{1}{3}u_1 D^{-1},$$

being in fact the recursion operator for this equation, see e.g. [13].

Thus, the requirements of Proposition 1 are met, and (9) with arbitrary (smooth) functions g and h has a degenerate formal symmetry of infinite rank and nonzero degree $\mathfrak{R} = \mathfrak{T}^{-1}\mathfrak{L}\mathfrak{T}$, where $\mathfrak{L} = \mathrm{diag}(\mathfrak{L}_1, \mathfrak{L}_2)$.

It would be interesting to find out under which conditions the system (9) has a *nondegenerate* formal symmetry of infinite rank and nonzero degree, and we intend to analyse this problem in more detail elsewhere.

A fairly straightforward but quite useful application of Proposition 1 is given by the following result.

Corollary 3 *Let $\vec{f}^\alpha = \sum_{i=0}^n a_i^\alpha(x)\vec{u}_i^\alpha$, $\alpha = 1, \dots, s-1$, where $a_i^\alpha(x)$ are $q_\alpha \times q_\alpha$ matrices. Denote for convenience $a_n^s = \partial\vec{f}^s/\partial\vec{u}_n^s$, and suppose that the matrices a_n^α and a_n^β have no common eigenvalues (i.e., the eigenvalues in question are distinct as functions) for all $\alpha \neq \beta$, $\alpha, \beta = 1, \dots, s$. Further assume that the evolution system $\vec{u}_t^s = \vec{f}^s$ possesses a formal symmetry $\mathfrak{L}_s$ of infinite rank and nonzero degree, and at least one of the matrices a_n^α is nonzero.*

Then the system (5) *with arbitrary (smooth) functions $\vec{h}^\alpha(x, t, \vec{u}^{\alpha+1}, \vec{u}_1^{\alpha+1}, \dots, \vec{u}_{n-1}^{\alpha+1}, \dots, \vec{u}^s, \vec{u}_1^s, \dots, \vec{u}_{n-1}^s)$, $\alpha = 1, \dots, s-1$, also possesses a formal symmetry of infinite rank and nonzero degree.*

Proof. We just take $\mathfrak{L}_\alpha = \mathfrak{F}_\alpha = \sum_{i=0}^n a_i^\alpha(x) D^i$ for $\alpha = 1, \dots, s-1$. □

3 Existence of nondegenerate formal symmetries

While applying the existence of formal symmetry of infinite rank as an integrability test one usually requires that the system in question should have a *nondegenerate* formal symmetry, cf. [11]. The results that follow provide easily verifiable sufficient conditions for the existence of formal symmetry with this property.

Corollary 4 *Under the assumptions of Corollary 3, suppose that the system $\vec{u}_t^s = \vec{f}^s$ has a* nondegenerate *formal symmetry $\mathfrak{L}_s$ of infinite rank and of nonzero degree k. Further assume that at least one of the following conditions holds:*

(i) $\det a_\alpha^n \neq 0$, $\alpha = 1, \dots, s-1$;

(ii) all matrices a_α^i, $\alpha = 1, \dots, s-1$, $i = 0, \dots, n$, are constant matrices;

(iii) one of the matrices a_α^n (say, a_δ^n) is degenerate: $\det a_\delta^n = 0$; $\det a_\alpha^n \neq 0$, *$\alpha \neq \delta$, $\alpha = 1, \dots, s-1$, and either a) there exists $m \in \mathbb{N}$ such that $m < n$ and we have $a_\delta^{m+1} = 0, \dots, a_\delta^n = 0$ while $a_\delta^m \neq 0$ and* $\det a_\delta^m \neq 0$, *or b) all matrices a_δ^j, $j = 0, \dots, n$, are constant ones.*

Then the system (5) *with arbitrary (smooth) functions $\vec{h}^\alpha(x, t, \vec{u}^{\alpha+1}, \vec{u}_1^{\alpha+1}, \dots, \vec{u}_{n-1}^{\alpha+1}, \dots, \vec{u}^s, \vec{u}_1^s, \dots, \vec{u}_{n-1}^s)$, $\alpha = 1, \dots, s-1$, possesses a* nondegenerate *formal symmetry $\mathfrak{R}$ of infinite rank and nonzero degree.*

Proof. In all cases we can represent $\mathfrak{R}$ in the form $\mathfrak{R} = \mathfrak{T}^{-1}\mathfrak{L}\mathfrak{T}$, where $\mathfrak{L}$ solves (6), and the nondegeneracy of $\mathfrak{L}$ clearly implies the same property for $\mathfrak{R}$. Therefore, it suffices to construct a nondegenerate solution $\mathfrak{L}$ of nonzero degree r for (6). We shall exhibit such solutions for all cases (i)–(iii). Their nondegeneracy will be obvious from the construction.

In the case (i) let r be the least common multiple of n and k, $\tilde{n} = r/n$, $\tilde{k} = r/k$, and we set $\mathfrak{L} = \mathrm{diag}(\mathfrak{F}_1^{\tilde{n}}, \dots, \mathfrak{F}_{s-1}^{\tilde{n}}, \mathfrak{L}_s^{\tilde{k}})$.

Likewise, in the case (ii) we set $\mathfrak{L} = \mathrm{diag}(\mathbf{1}_{q_1} D^k, \dots, \mathbf{1}_{q_{s-1}} D^k, \mathfrak{L}_s)$, where $\mathbf{1}_{q_\alpha}$ is $q_\alpha \times q_\alpha$ unit matrix.

In the case (iii, a) let r be the least common multiple of n, m and k, and we set $\mathfrak{L} = \mathrm{diag}(\mathfrak{F}_1^{\tilde{n}}, \dots, \mathfrak{F}_{\delta-1}^{\tilde{n}}, \mathfrak{F}_\delta^{\tilde{m}}, \mathfrak{F}_{\delta+1}^{\tilde{n}}, \dots, \mathfrak{F}_{s-1}^{\tilde{n}}, \mathfrak{L}_s^{\tilde{k}})$, where $\tilde{n} = r/n$, $\tilde{m} = r/m$, $\tilde{k} = r/k$.

Finally, in the case (iii, b), taking for r the least common multiple of n and k, we set $\mathfrak{L} = \mathrm{diag}(\mathfrak{F}_1^{\tilde{n}}, \dots, \mathfrak{F}_{\delta-1}^{\tilde{n}}, \mathbf{1}_{q_\delta} D^r, \mathfrak{F}_{\delta+1}^{\tilde{n}}, \dots, \mathfrak{F}_{s-1}^{\tilde{n}}, \mathfrak{L}_s^{\tilde{k}})$, where $\mathbf{1}_{q_\delta}$ is $q_\delta \times q_\delta$ unit matrix, $\tilde{n} = r/n$, $\tilde{k} = r/k$. □

Corollary 5 *Let $\vec{f}^\alpha = \sum_{i=0}^n a_i^\alpha(t)\vec{u}_i^\alpha$, $\alpha = 1, \dots, s-1$, where $a_i^\alpha(t)$ are $q_\alpha \times q_\alpha$ matrices. Again denote for convenience $a_n^s = \partial\vec{f}^s/\partial\vec{u}_n^s$, and suppose that the matrices a_n^α and a_n^β have no common eigenvalues, i.e., the eigenvalues in question are distinct as functions, for all $\alpha \neq \beta$, $\alpha, \beta = 1, \dots, s$, and at least one of the matrices a_n^α is nonzero.*

Then the system (5) *with arbitrary smooth functions $\vec{h}^\alpha(x, t, \vec{u}^{\alpha+1}, \vec{u}_1^{\alpha+1}, \dots, \vec{u}_{n-1}^{\alpha+1}, \dots, \vec{u}^s, \vec{u}_1^s, \dots, \vec{u}_{n-1}^s)$, $\alpha = 1, \dots, s-1$, possesses a (nondegenerate) formal symmetry $\mathfrak{L}_s$ of infinite rank and of nonzero degree k, if so does the system $\vec{u}_t^s = \vec{f}^s$.*

Proof. Let $\mathfrak{L} = \mathrm{diag}(\mathbf{1}_{q_1} D^k, \dots, \mathbf{1}_{q_{s-1}} D^k, \mathfrak{L}_s)$ and $\mathfrak{T}$ be a formal series constructed in Lemma 2. Then $\mathfrak{R} = \mathfrak{T}^{-1}\mathfrak{L}\mathfrak{T}$ is a formal symmetry of infinite rank and of degree $k \neq 0$ for (5). Finally, if $\mathfrak{L}_s$ is nondegenerate, then so does $\mathfrak{R}$. □

For instance, the Bakirov system (1) and the system (2), investigated by Sanders and van der Kamp [8], indeed meet the requirements of Corollaries 3, 4 and 5 and therefore have nondegenerate formal symmetries of infinite rank and nonzero degree. What is more, by Corollary 5 any system of the form

$$u_t = a(t)u_n + K(x, t, v, v_1, \dots, v_{n-1}), v_t = b(t)v_n$$

has a nondegenerate formal symmetry of infinite rank and nonzero degree, provided $a(t) \neq b(t)$.

Following Kupershmidt [9], consider a system $\vec{u}_t = \vec{F}(x, t, \vec{u}, \dots, \vec{u}_n)$ with $n \geq 2$ and $\det \partial\vec{F}/\partial\vec{u}_n \neq 0$, and its (vectorial) *logarithmic extension*

$$\vec{u}_t = \vec{F}(x, t, \vec{u}, \dots, \vec{u}_n), \vec{v}_t = \vec{G}(x, t, \vec{u}, \dots, \vec{u}_{n-1}). \tag{10}$$

Here $\vec{u}$ and $\vec{v}$ are q_1- and q_2-component vectors, respectively, and $\vec{G}$ is an arbitrary (smooth) q_2-component vector function.

By Corollary 5 the system (10) possesses a (nondegenerate) formal symmetry of infinite rank and nonzero degree if so does $\vec{u}_t = \vec{F}(x, t, \vec{u}, \dots, \vec{u}_n)$. This fact suggests that, in addition to the four types of extensions of integrable systems, introduced in [9], it is natural to consider the fifth one, namely, the extensions which "inherit" (nondegenerate) formal symmetry from the original system.

4 Conclusions and discussion

We have shown above that a fairly large class of evolution systems (5) has a (nondegenerate) formal symmetry of infinite rank and nonzero degree, provided so do all "building blocks" of (5), that is, $\vec{u}^\alpha_t = \vec{f}^\alpha$, and the coefficients of formal symmetries of the first $s-1$ blocks depend on x and t only. In other words, under certain conditions the system (5) inherits some of integrability properties of its blocks.

Let us also mention that once a solution $\vec{u}^s(x, t)$ of $\vec{u}^s_t = \vec{f}^s$ is known, recovering the corresponding solution of (5) amounts to solving linear inhomogeneous PDEs, provided $\vec{f}^\alpha$, $\alpha = 1, \dots, s-1$ are linear in $\vec{u}^\alpha_j$. In this case, if the system $\vec{u}^s_t = \vec{f}^s$ is exactly solvable, then the same is true for (5). However, as show the examples of the Bakirov system [1], and of the systems constructed by Sanders and van der Kamp in [8], if the system $\vec{u}^s_t = \vec{f}^s$ has infinitely many non-Lie-point local generalized symmetries, the system (5) does not necessarily have the same property even if it possesses a formal symmetry of infinite rank and nonzero degree. We encounter here an intriguing phenomenon of 'disappearing' of symmetries, which, surprisingly, is due to some subtle number-theoretical effects [2, 8].

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